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A COMPARISON OF NEWTON-LIKE METHODS FOR THE TRANSONIC SMALL DIS--ETC(U)

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A COMPARISON OF NEWTON-LIKE METHODS FOR THE TRANSONIC SMALL DISTURBANCE EQUATION

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BY A. B. STEPHENS A. G. WERSCHULZ

RESEARCH AND TECHNOLOGY DEPARTMENT

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We investigate the efficiency of Newton's method and two variants of Newton's method for the numerical solution of the small disturbance equation of non-lifting transonic flow past a parabolic airfoil. In particular, it is shown that the efficiency of Newton's method can be substantially improved with respect to both storage and computational time if the Jacobian matrix is suitably altered.

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FOREWORD

This report summarizes an investigation of three Newton-like methods for the numerical solution of the small disturbance equation of non-lifting transonic flow past a parabolic airfoil. This work was performed by the authors, who are members of the Applied Mathematics Branch of NSWC. The authors wish to thank Dr. Gregory R. Shubin of NSWC for his interest in the work reported herein, thanking him for his many comments and suggestions.

B. F. DESAVAGE
By direction

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ILLUSTRATIONS

1. INTRODUCTION

In this paper we investigate the effectiveness of Newton's method and two variants of Newton's method for solving the nonlinear difference equations that approximate the transonic small disturbance equation. In particular, we consider the case of non-lifting flow over a parabolic airfoil. Numerical methods for solving both the small disturbance and the full potential equations have been the object of a good deal of research in the last few years. The references [1] - [10] give an idea of the range of different methods that have appeared in the literature.

Although Newton's method converges quadratically, requiring fewer iterations than linearly-convergent methods, its effectiveness for nonlinear problems with many unknowns is open to question for several reasons. First, at each iteration one must typically solve a large system of linear equations. The relative success of Newton's method depends in part on how effectively one can solve the linear system either directly or iteratively. A direct method may require an excessive amount of computer memory, and require out-of-core solvers, and for

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1. Bailey, F. R., "On the Computation of Two- and Three-dimensional Steady Transonic Flows by Relaxation Methods," Lecture Notes in Physics, v. 41: Progress in Numerical Fluid Dynamics (H. J. Wirz, ed.), Springer-Verlag, Berlin, pp. 1-77, 1975.
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problems of the type we are considering an iterative method such as SOR may fail to converge. Secondly, the domain of convergence for Newton's method may be relatively small, requiring a good initial guess for convergence.

The two variants of Newton's method we consider seem to be more robust than Newton's method with respect to convergence and require less storage; moreover, one requires less computer time to converge.

In section 2 we discuss the formulation of the transonic small disturbance equation and the Murman-Cole difference scheme for its approximation. In section 3 we discuss the two variants of Newton's method for the nonlinear difference equations. One variant is a line-Newton successive over-relaxation (LINSOR) method and the other uses only the lower triangular portion of the Jacobian with over-relaxation of the iterates (LONSOR).

The numerical performance of these methods and Newton's method are compared in section 4. Briefly, these numerical results indicate, for example, that with the proper choice of an over-relaxation parameter, the LINSOR method, when compared to Newton's method is faster, needs a good deal less storage, and converges for a wider range of starting guesses. The LONSOR method also needs less storage and on a 25 x 25 mesh satisfies the steady state error criterion in a time comparable to Newton's method.

2. FORMULATION OF THE PROBLEM

We consider the two dimensional small disturbance equation

$$S(\phi) = (K-(\gamma+1)\phi_x)\phi_{xx} + \phi_{yy} = 0 \tag{2.1}$$

which models inviscid flow over a thin wing. Here ϕ is the potential function of the perturbation from the free stream velocity. The transonic similarity parameter K is defined by

$$K = (1-M_\infty^2)/\delta^{2/3}$$

where M_∞ is the free stream Mach number, assumed to be close to 1, and δ is the airfoil thickness ratio. For a derivation of (2.1) from the Euler equations see, for example, [11].

BOUNDARY CONDITIONS. In order to approximate boundary conditions at infinity the airfoil is enclosed within a sufficiently large rectangle R. For

11. Nahamira, S., Computational Methods in Engineering and Science, Wiley-Interscience, New York, 1977.

lifting flows, ϕ at the boundary ∂R of R is defined in terms of the circulation. For nonlifting flows we set $\phi = 0$ on ∂R.

At the boundary of the airfoil corresponding to a point (x, 0) it is required that $\phi_y(x, 0^+) = F'_+(x)$ and $\phi_y(x, 0^-) = F'_-(x)$ where $F_+(x)$ and $F_-(x)$ describe the upper and lower boundaries of the airfoil respectively. We consider a parabolic airfoil where $F_+(x)$ and $F_-(x)$ are given by $2\delta x(1-x)$ and $-2\delta x(1-x)$ respectively. These conditions are applied on the mean surface y = 0 and approximate the condition that the flow at the airfoil be tangent to the airfoil.

FLOW OF MIXED TYPE. Equation (2.1) together with the boundary conditions constitute a nonlinear boundary value problem for ϕ. Basically the type of flow at a given point is regulated by the sign of $K - (\gamma+1)\phi_x$. For a given δ when M is sufficiently close to, but less than, one, $K - (\gamma+1)\phi_x$ is negative in certain regions above and below the airfoil. In these regions the flow is supersonic and is bounded by a sonic line and weak shock. Outside this region $K - (\gamma+1)\phi_x$ is positive and the flow is subsonic.

Different numerical techniques have been devised to approximate the solution ϕ. We now describe a finite difference scheme and iterative methods for its solution.

FINITE DIFFERENCE EQUATIONS. A prevalent method for discretizing (2.1) is the Murman-Cole difference scheme introduced in [12] and later modified to more accurately capture shocks [13]. A more recent modification which excludes expansion shocks is due to Enquist and Osher [2, 3].

In the Murman-Cole scheme directionality is introduced by using centered differences for ϕ_x and ϕ_{xx} in the subsonic region and backward differences in the supersonic region. This procedure is implemented as follows. We partition the rectangle R with a uniform mesh $x_i = ih$, $0 \le i \le n+1$, $y_j = j\tilde{h}$, $0 \le j \le m+1$. We define $\phi_{ij}^{(k)}$ to be the k^{th} approximation to $\phi(x_i, y_j)$ defined by some iterative scheme. At the mesh point (x_i, y_j) define

12. Murman, E. M. and Cole, J. D., "Calculation of Plane Steady Transonic Flows," AIAA Journal, v. 9, no. 1, pp. 114-121, 1971.
13. Murman, E. M., "Analysis of Embedded Shock Waves Calculated by Relaxation Methods," Proceedings of AIAA Computational Fluid Dynamics Conference, Palm Springs, Ca., pp. 27-40, July 1973.

$$E_{ij} = K - (\gamma+1)\left(\phi_{i+1,j}^{(k-1)} - \phi_{i-1,j}^{(k)}\right)/2\Delta x$$

and

$$H_{ij} = K - (\gamma+1)\left(\phi_{ij}^{(k-1)} - \phi_{i-2,j}^{(k)}\right)/2\Delta x$$

then E_{ij} and H_{ij} are discrete versions of the term $K - (\gamma+1)\phi_x$. If $E_{ij} > 0$, (2.1) is judged to be elliptic at (x_i, y_j) and is differenced

$$E_{ij}\delta_x^2\phi_{ij} + \delta_y^2\phi_{ij} = 0$$

where

$$\delta_x^2\phi_{ij} = (\phi_{i+1,j} - 2\phi_{ij} + \phi_{i-1,j})/(\Delta x)^2$$

and

$$\delta_y^2\phi_{ij} = (\phi_{i,j+1} - 2\phi_{ij} + \phi_{i,j-1})/(\Delta y)^2.$$

If $E_{ij} < 0$ and $H_{ij} < 0$ (2.1) is hyperbolic at (x_i, y_j) and differenced as

$$H_{ij}\delta_x^2\phi_{i-1,j} + \delta_y^2\phi_{ij} = 0.$$

If $E_{ij} < 0$ and $H_{ij} > 0$ the point (x_i, y_j) is regarded as lying on a sonic line and (2.1) at (x_i, y_j) is differenced as

$$\delta_y^2\phi_{ij} = 0.$$

In the case that $E_{ij} > 0$ and $H_{ij} < 0$ the point (x_i, y_j) lies at a shock and the original Murman-Cole formulation was to use centered differencing. Later this was modified to incorporate a "shock point" operator given by

$$(E_{ij} + H_{ij})\delta_x^2\phi_{ij} + \delta_y^2\phi_{ij} = 0.$$

In this paper we use centered differencing at a shock point since we are not so interested in the accuracy of the shock strength or shock location but rather in the efficient solution of the finite difference equations. Finite difference approximations for the specification of ϕ_y on the airfoil boundary are actually applied on the rows of mesh points (x_i, y_j) immediately above and below the mean surface of the airfoil. There the given expression for ϕ_y is incorporated by

$$\phi_{yy} \cong \frac{1}{\Delta y}\left[\frac{(\phi_{i,j+2} - \phi_{i,j+1})}{\Delta y} - \phi_y\Big|_{y=0}\right]$$

above the wing and

$$\phi_{yy} \cong \frac{1}{\Delta y}\left[\phi_y\Big|_{y=0} - \frac{(\phi_{i,j-1} - \phi_{i,j-2})}{\Delta y}\right]$$

below the wing.

3. ITERATIVE METHODS

We write the nonlinear $mn \times mn$ system of equations as $S(\bar{\phi}) = 0$. A general iterative scheme for the solution of $S(\bar{\phi}) = 0$ is

$$N(\bar{\phi}^k)(\bar{\phi}^{k+1} - \bar{\phi}^k) = -\alpha S(\bar{\phi}^k) \tag{3.1}$$

where $N(\bar{\phi}^k)$ is an $mn \times mn$ matrix and α is a relaxation parameter. This iterative method may be viewed as a discretization of the time-dependent equation

$$N(\phi(t))\dot{\phi}(t) + S(\phi(t)) = 0. \tag{3.2}$$

In the case that N is the Jacobian matrix J whose $(i,j)^{\underline{th}}$ entry is given by the derivative of the $i^{\underline{th}}$ equation with respect to the $j^{\underline{th}}$ unknown, the iterative scheme (3.1) is a damped Newton's method for the system (3.2); in this case, J has the block form

$$J = \begin{bmatrix} D_1 & E_1 & & & & & \\ C_2 & D_2 & E_2 & & & 0 & \\ B_3 & C_3 & D_3 & E_3 & & & \\ & B_4 & C_4 & D_4 & E_4 & & \\ 0 & & \ddots & \ddots & \ddots & \ddots & \\ & & & B_{n-1} & C_{n-1} & D_{n-1} & E_{n-1} \\ & & & & B_n & C_n & D_n \end{bmatrix}.$$

The n blocks B_i, C_i, D_i, E_i are $m \times m$ matrices. The matrix D_i is tridiagonal and the blocks B_i, C_i, E_i are diagonal. The blocks B_i have non-zero entries corresponding to points (x_i, y_j) in the hyperbolic region.

LINE-NEWTON SOR (LINSOR). Despite the rapid convergence of Newton's method for good initial guesses, it suffers from two major drawbacks: its storage requirement of $O(m^2n)$ entries, due to the fill-in which occurs when Gaussian elimination is used to solve the system (3.1) with N = J, and its operation count

of $O(m^3n)$ operations per iteration. In order to facilitate the solution of (3.1) by Gaussian elimination, we let N be the block lower triangular matrix J* which is formed by deleting the blocks E_i from the matrix J. If the diagonal blocks D_i are nonsingular, we may then solve (3.1) as a series of block solves: writing

$$\bar{\phi}^{k+1} - \bar{\phi}^{k} = \bar{e}^{k} = (e_1^T \cdots e_n^T)^T$$

and

$$S(\bar{\phi}^{k}) = (f_1^T \cdots f_n^T)^T,$$

we have a series of n tridiagonal systems

$$D_1 e_1 = -f_1$$

$$D_i e_i = -f_i - C_i e_{i-1} \quad (2 \leq i \leq n).$$

In order to accelerate convergence, we overwrite $\bar{\phi}^{k+1}$ by a convex combination of the old and new iterates

$$\phi_i^{k+1} \leftarrow \omega_i \phi_i^{k+1} + (1-\omega_i)\phi_i^{k}.$$

Here the relaxation parameter ω_i satisfies $\omega_i > 1$ for those components of $\bar{\phi}^{k+1}$ lying in the supersonic region and $\omega_i = 1$ for those components in the sonic and subsonic regions. Note that this method requires storage O(m) and uses O(mn) operations per iteration, so that an iteration of the LINSOR method is far cheaper than an iteration of Newton's method and the LINSOR method has a much smaller storage overhead than does Newton's method.

One may show that the method above is equivalent to the following. We consider the difference equations approximating (1.1) and choose a column ℓ in the finite-difference mesh. We consider the values $\phi_{\ell 1},\cdots,\phi_{\ell m}$ on column ℓ as the only unknowns and consider only those difference equations which involve the $\phi_{\ell j}$. In these equations, we use the most recent values of $\phi_{\ell-1,j}$ and $\phi_{\ell+1,j}$ when they are needed. One iteration of Newton's method is used to approximately solve for the unknowns $\phi_{\ell j}$. These values are then relaxed where appropriate and updated. We then move to column $\ell+1$ and repeat the process. Starting at column 1 and continuing to column n constitutes a sweep. The grid is repeatedly swept until steady state is reached. This technique may be viewed as a type of line SOR-Newton sweep.

LOWER NEWTON SOR (LONSOR). We may also construct a new iteration matrix N by modifying J in the following simple manner: we eliminate from J all entries above the diagonal, leaving a lower triangular matrix with nonzero elements on the diagonal. We may quickly solve the resulting system using forward substitution. Those ϕ_{ij} which are in the elliptic region of flow can be relaxed as soon as they are computed. The matrix N can now be regarded as the Jacobian matrix corresponding to the equations (3.1), where at the point (x_i, y_j), $\phi_{i+1,j}$ and $\phi_{i,j-1}$ are no longer treated as unknowns, but instead their most recent values are substituted.

The numerical results for Newton's method, LINSOR, and LONSOR are presented in the next section.

4. RESULTS

In this section, we present computational results for the problem stated in Section 3. The computational region was the rectangle $[-1.5, 1.5] \times [-1.0, 1.0]$ of the x-y plane, the wing being a parabolic airfoil on the segment [0, 1] of the x-axis. The physical parameters used were $M_\infty = 0.9$ for the free-stream Mach number and δ between 0.05 and 0.065 for the airfoil thickness ratio. The problem was run on uniform 20×20 and 25×25 meshes of unknowns. The error was measured by the maximum modulus of the residual vector, and the stopping criterion was that the error be less than 10^{-4}.

The three methods that were successfully tested are the Line-Newton SOR (LINSOR) method, the Lower-Newton SOR (LONSOR) method, and the full Newton method. For the results reported in this section an initial guess of $\phi = 0$ was used for all three methods.

The LINSOR and LONSOR methods require an iteration parameter ω. It was decided to use over-relaxation only on the elliptic region, choosing $\omega = 1$ in the hyperbolic and parabolic regions. In Figure 1, we present experimental evidence which shows that choosing $\omega = 1.65$ in the elliptic region is close to optimal for the 20×20 mesh with the LINSOR method; it was similarly determined that choosing $\omega = 1.85$ in the elliptic region is close to optimal for the 20×20 mesh with the LONSOR method. Moreover, the dependence of ω on mesh size does not appear to be too critical; the same values were used successfully on the 25×25 mesh.

Figure 1 indicates that elliptic-region over-relaxation successfully accelerates convergence. However, it is not a priori clear at which iteration the over-relaxation should start. For example, when over-relaxation was started

immediately for the case $M_\infty = 0.95$, the method diverged; experimentally it was found that for $M_\infty = 0.95$, one could successfully start over-relaxation after about twelve iterations. The shift from the non-over-relaxed to the over-relaxed scheme causes the error to increase momentarily.

Although the LINSOR and LONSOR methods were relatively simple to implement, there was a storage problem with the full Newton method. This problem was due to the fill-in of the factorization of the sparse Jacobian matrix as the product of lower- and upper-triangular banded matrices. To alleviate this problem, the Yale sparse matrix package (YSMP) with uncompressed storage option for nonsymmetric matrices [14] was used. Despite the improvement in storage when using YSMP instead of a banded solver, the full Newton/YSMP program required a prohibitive amount of storage as the number of unknowns increased. For instance, the Newton/YSMP program could barely fit the 25×25 problem onto a CDC 6500, while LINSOR was able to handle problems of size 40×40 or larger.

In Figures 2 and 3, results are given for the 20×20 and 25×25 meshes, respectively. In each figure, the maximum modulus of the residual vector is plotted as a function of the CP time used, for each of the three methods tested. One sees that LINSOR has the fastest convergence in each case, meeting the 10^{-4} error criterion for the 20×20 and 25×25 matrices in 9.229 and 22.06 seconds, respectively. The full Newton method was next fastest, requiring 15.37 and 31.62 seconds for the two meshes. LONSOR was the slowest of the three, requiring 18.74 and 35.01 seconds.

Not surprisingly, only a few full Newton iterations were needed to solve the problem to within 10^{-4} (6 and 7 iterations for the 20×20 and 25×25 cases, respectively). However, the time required to solve the linear system resulting from each Newton iteration, combined with the space required to store the triangular factors of the coefficient matrix, rendered the full Newton method inferior to LINSOR.

A number of other methods were attempted. Following ideas of Jameson [5], a fast Poisson solver was used after every few iterations of the LINSOR method; this

14. Eisenstadt, S. C., Gursky, M. C., Schultz, M. H., and Sherman, A. H., "Yale Sparse Matrix Package," Yale University Computer Science Department Research Report #114, New Haven, 1977.

was found to be difficult to implement effectively. The SOR method, when applied to the linear system arising from Newton's method, was found to be "non-robust" in the sense that it failed to work for $M_\infty = 0.95$. (See [15] for discussion of using the SOR method for mixed problems.)

The block-SOR and conjugate gradient methods were also used in the attempt to solve the linear system arising from Newton's method. The former sometimes showed oscillatory behavior or diverged, while the latter required a prohibitively large number of conjugate gradient iterations to solve the linear system with sufficient accuracy.

It is interesting to note that using different methods to solve the Murman-Cole difference equations may yield different solutions. Using a relatively thick wing at $M_\infty = 0.95$, and a 10×10 grid of unknowns, LINSOR found a solution whose sonic line was uniformly one mesh point to the right of that found by the full Newton method (see Figure 4). Since the sonic lines were different, the type-dependence of the difference scheme implies that the two final systems of nonlinear equations were different, and had different solutions.

Finally, we remark that using LINSOR and a continuation technique with respect to M_∞, it is possible to compute solutions up to and past Mach 1 for the small disturbance equation.

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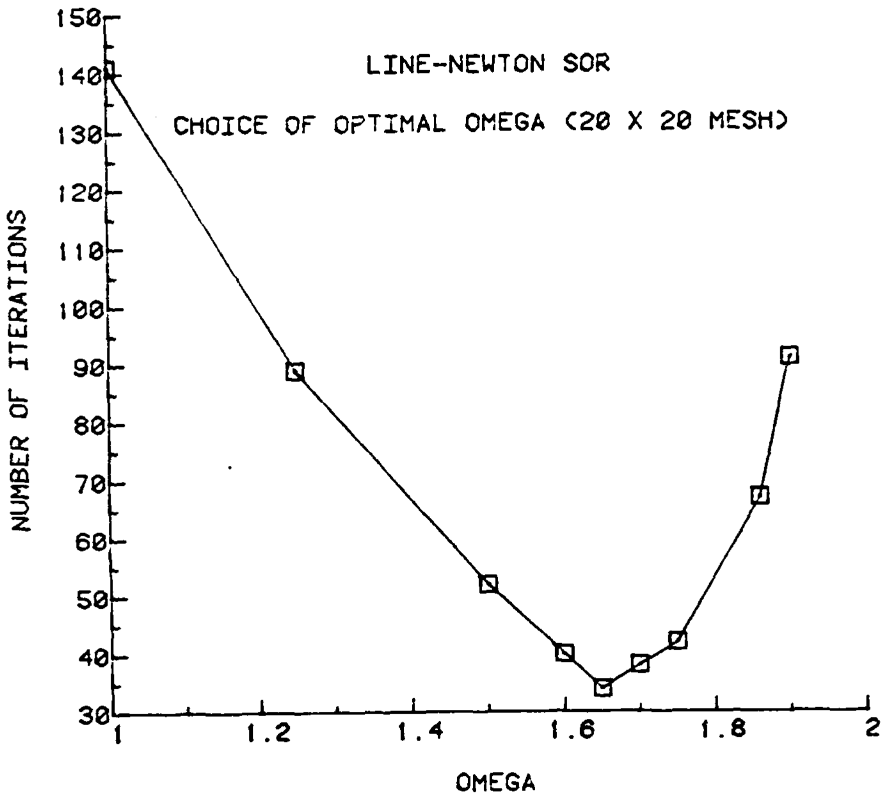

FIGURE 1 CHOICE OF OPTIMAL ω FOR THE LINSOR METHOD USING A 20×20 MESH

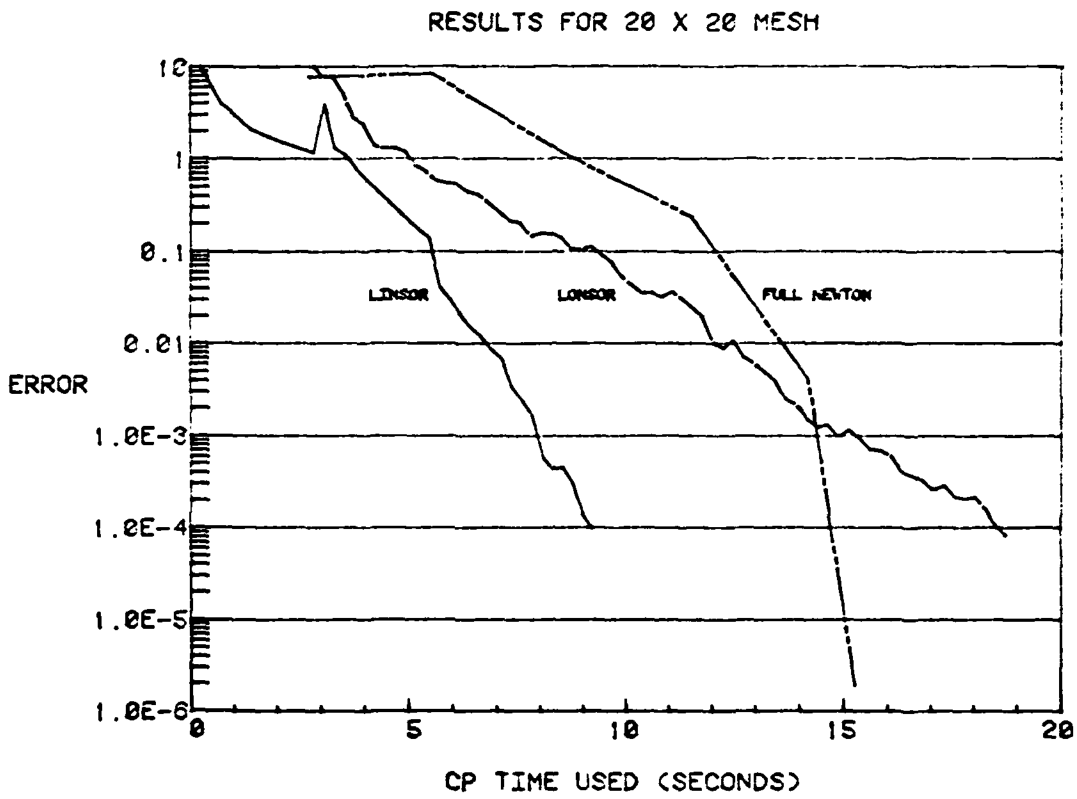

FIGURE 2 ERROR AS A FUNCTION OF CENTRAL PROCESSOR TIME USED ON A 20×20 MESH

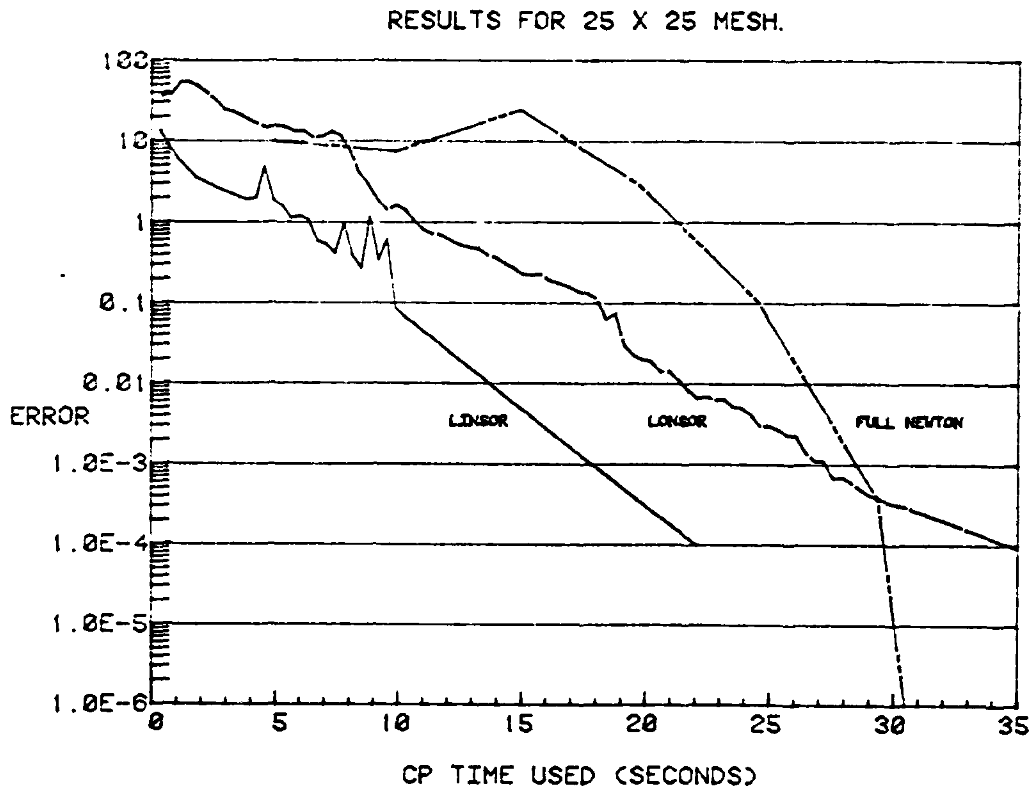

FIGURE 3 ERROR AS A FUNCTION OF CENTRAL PROCESSOR TIME USED ON A 25×25 MESH

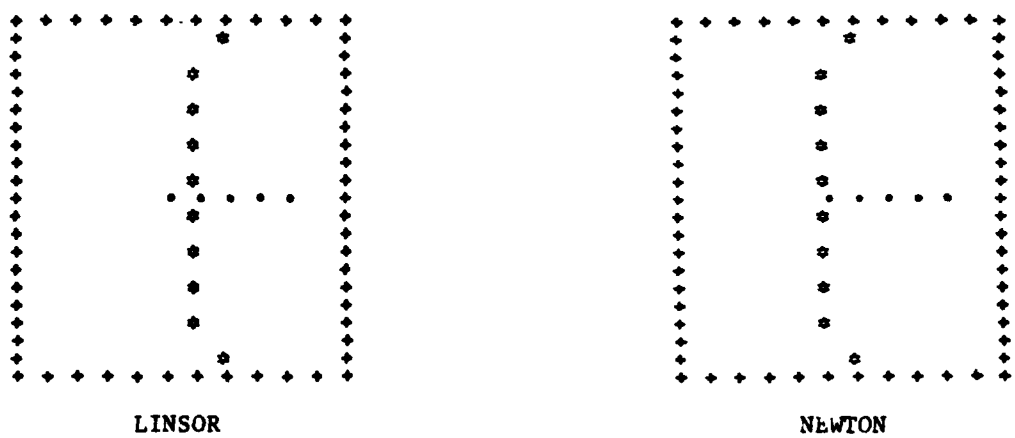

FIGURE 4 SONIC PROFILES OF DIFFERENT SOLUTIONS FOUND BY THE LINSOR AND NEWTON METHODS ON A 10×10 MESH

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